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Functional relations and nested Bethe ansatz for $sl(3)$ chiral Potts model at $q^2 = -1$

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Abstract. We obtain the functional relations for the eigenvalues of the transfer matrix of the $sl(3)$ chiral Potts model for $q^2 = -1$. For the homogeneous model in both directions a solution of these functional relations can be written in terms of roots of Bethe ansatz-like equations. In addition, a direct nested Bethe ansatz has also been developed for this case.

1. Introduction

The discovery of the chiral Potts model (CPM) and new N -state solutions of the Yang–Baxter relation [1–4] is one of the most impressive results in the theory of two-dimensional lattice integrable systems [5]. A remarkable feature of this model is that the spectral parameters belong to a high genus algebraic curve. The reason for this remained unclear until it was shown that the weights of the CPM naturally appear as intertwiners of the cyclic representations for $sl(2)$ L -operators [6] related to the six-vertex model [7].

The next step was to generalize the CPM for the $sl(n)$ case for $q^N = -1$ [8]. As a result, a new N^{n-1} -state family of generalized CPMs has been obtained with Boltzmann weights satisfying the Yang–Baxter relation [9, 10].

Then Bazhanov and Baxter [11] made a remarkable observation that the $sl(n)$ CPM with $N = 2$ is related to the integrable three-dimensional Zamolodchikov model [12, 13]. Therefore, a new link between two quite different regions of the theory of integrable models has been revealed. It is well known that the tetrahedron equations provide sufficient conditions for a commutativity of the transfer matrices in three dimensions [14, 15]. As a result, a new N -state three-dimensional family of integrable models has been discovered with Boltzmann weights satisfying the tetrahedron equation [16, 17].

The Zamolodchikov model attracted a lot of attention even before the appearance of the CPM. Its partition function in the thermodynamic limit has been calculated in [18] and the related Hamiltonian has been studied in [19]. It was shown in [19] that the Zamolodchikov model with two layers is equivalent to the critical two-dimensional free-fermion model. However, the structure of the spectrum of the Zamolodchikov model with three and more layers remains unclear.

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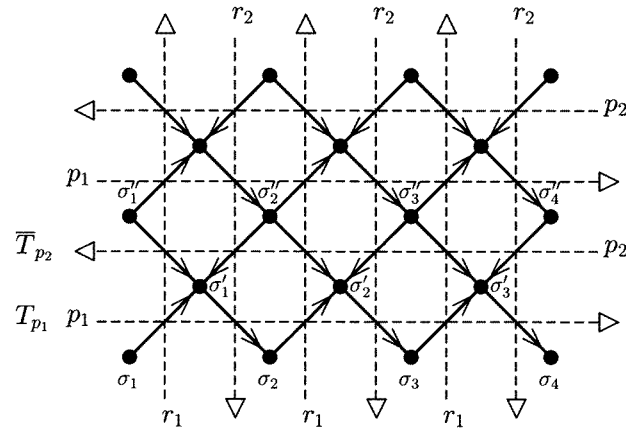


Figure 1. The square lattice \mathcal{L} .

In this paper we try to take several steps towards the understanding of the structure of the spectrum of the $sl(3)$ CPM which is equivalent to the ‘modified’ (in the terminology of [11, 18]) three-layer Zamolodchikov model. Starting from the general case we carefully analyse the homogeneous case of the $sl(3)$ CPM where a usual nested Bethe ansatz technique [20] can be applied.

The paper is organized as follows. In section 2 we give a full description of the $sl(3)$ CPM and introduce all necessary notations. In section 3 we describe a ‘fusion’ procedure [21–23] for the $sl(3)$ CPM for $q^2 = -1$ ($N = 2$) and obtain the closed functional equations for eigenvalues of the transfer matrix. Section 4 is devoted to the spectrum of the model in the completely homogeneous case. In section 5 we apply the nested Bethe ansatz technique to obtain eigenvalues of the transfer matrix for the homogeneous case. In the last section we discuss our results and further directions for investigation.

2. The $sl(3)$ CPM

In this section we give all necessary definitions for the $sl(3)$ CPM with fixed N . All definitions and formulae of this section are just a specification of the $sl(n)$ case for $n = 3$ [10].

Consider an oriented square lattice \mathcal{L} and its medial lattice \mathcal{L}' (shown in figure 1 by solid and dashed lines, respectively). The oriented vertical (horizontal) lines of \mathcal{L}' carry rapidity variables p_1, p_2 (r_1, r_2) in an alternating order (note that the orientations of rapidity lines shown by open arrows alternate too). The edges of the lattice \mathcal{L} are oriented in such a way that all the NW–SE edges have the same (NW–SE) direction while the NE–SW edges are oriented in a checkerboard order.

Each rapidity variable is represented by three 2-vectors $(h_i^+(p), h_i^-(p)), i = 1, 2, 3$ which specify the point p of the algebraic curve Γ defined by relations

$$\begin{pmatrix} h_i^+(p)^N \\ h_i^-(p)^N \end{pmatrix} = K_{ij} \begin{pmatrix} h_j^+(p)^N \\ h_j^-(p)^N \end{pmatrix} \quad \forall i, j = 1, 2, 3 \tag{2.1}$$

where K_{ij} are 2×2 complex matrices of moduli satisfying relations

$$\det K_{ij} = 1 \quad K_{ii} = K_{ij}K_{jk}K_{ki} = 1. \tag{2.2}$$

Hereafter we imply that indices i, j, k, \dots run the values 1, 2, 3 modulo 3.

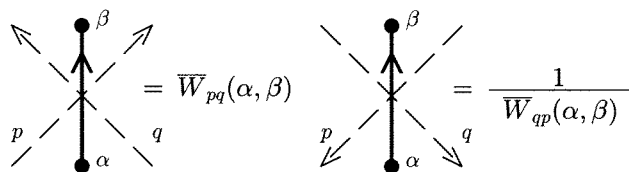


Figure 2. The weights $\bar{W}_{pq}(\alpha, \beta)$ and $\bar{W}_{pq}^{-1}(\alpha, \beta)$.

Below we need the automorphism τ on the curve Γ defined as follows:

$$h_j^+(\tau(p)) = h_j^+(p) \quad h_j^-(\tau(p)) = \omega h_j^-(p) \quad j = 1, 2, 3. \quad (2.3)$$

On each site of the lattice \mathcal{L} place $Z_N \times Z_N$ spins, which are described by a local variable

$$\alpha = (\alpha_1, \alpha_2) \quad \alpha_1, \alpha_2 = 0, 1, \dots, N-1. \quad (2.4)$$

There are two kinds of neighbouring local state pairs depending on the relative orientation of the dashed and solid lines as indicated in figure 2, with states α and β , and Boltzmann weights $\bar{W}_{pq}(\alpha, \beta)$ and $(\bar{W}_{qp}(\alpha, \beta))^{-1}$ on the edges of \mathcal{L} .

The function $\bar{W}_{pq}(\alpha, \beta)$, $\alpha, \beta \in Z_N \times Z_N$ is defined by the following relations:

$$\bar{W}_{p,q}(\alpha, \beta) = \omega^{Q(\alpha, \beta)} g_{pq}(0, \alpha - \beta) \quad \omega = \exp\left(\frac{2\pi i}{N}\right) \quad (2.5)$$

where

$$Q(\alpha, \beta) = \beta_1(\beta_1 - \alpha_1) + \beta_2(\beta_1 - \alpha_1 + \beta_2 - \alpha_2), \quad \alpha, \beta \in Z_N \times Z_N \quad (2.6)$$

and the function $g_{pq}(0, \alpha)$ has the following form:

$$g_{pq}(0, \alpha) = \frac{\prod_{\beta=0}^{\alpha_1+\alpha_2-1} (h_0^+(p)h_0^-(q) - h_0^+(q)h_0^-(p)\omega^{-\beta})}{\prod_{i=1}^2 \prod_{\beta_i=0}^{\alpha_i-1} (h_i^+(p)h_i^-(q) - h_i^+(q)h_i^-(p)\omega^{1+\beta_i})}. \quad (2.7)$$

We choose a normalization of $\bar{W}_{pq}(\alpha, \beta)$ as

$$\bar{W}_{pq}(0, 0) = 1. \quad (2.8)$$

Then it is easy to see that

$$\bar{W}_{pp}(\alpha, \beta) = \bar{\delta}_{\alpha, \beta} \quad \bar{\delta}_{\alpha, \beta} \equiv \begin{cases} 1 & \alpha = \beta \pmod{N} \\ 0 & \text{otherwise.} \end{cases} \quad (2.9)$$

The function $\bar{W}_{pq}(\alpha, \beta)$ satisfies the inversion relation

$$\sum_{\beta \in Z_N \times Z_N} \bar{W}_{pq}(\alpha, \beta) \bar{W}_{qp}(\beta, \gamma) = \bar{\delta}_{\alpha, \gamma} \Phi_{pq} \quad (2.10)$$

where

$$\Phi_{pq} = N^2 \frac{x_p^N - x_q^N}{x_p - x_q} \prod_{i=1}^3 \frac{x_i(p) - x_i(q)}{x_i(p)^N - x_i(q)^N} \quad x_i(p) \equiv \frac{h_i^-(p)}{h_i^+(p)} \quad x_p \equiv \prod_{i=1}^3 x_i(p). \quad (2.11)$$

Now let us suppose that our lattice \mathcal{L} has M sites in a horizontal direction and L sites in a vertical one (L should be even). As usual we imply cyclic boundary conditions in both directions.

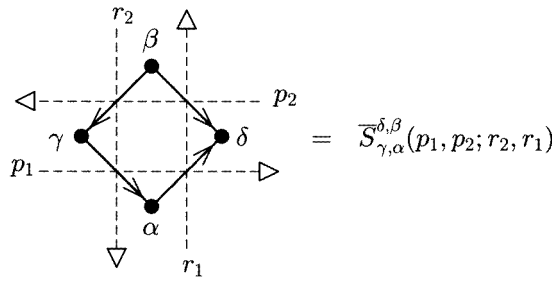


Figure 3. R -matrix $\bar{S}_{\gamma, \alpha}^{\delta, \beta}(p_1, p_2; r_1, r_2)$.

Let us denote the spin variables of three consecutive rows as $\sigma_1, \dots, \sigma_M, \sigma'_1, \dots, \sigma'_M$ and $\sigma''_1, \dots, \sigma''_M$ (see figure 1). Then we can define two $N^{2M} \times N^{2M}$ transfer matrices T_{p_1} and \bar{T}_{p_2} of the length M

$$[T_{p_1}]_{\sigma_1 \dots \sigma_M}^{\sigma'_1 \dots \sigma'_M} = \prod_{i=1}^M \bar{W}_{p_1 r_1}(\sigma_i, \sigma'_i) \bar{W}_{r_2 p_1}(\sigma'_i, \sigma_{i+1}) \tag{2.12}$$

$$[\bar{T}_{p_2}]_{\sigma'_1 \dots \sigma'_M}^{\sigma''_1 \dots \sigma''_M} = \prod_{i=1}^M \frac{\bar{W}_{p_2 r_2}(\sigma''_{i+1}, \sigma'_i)}{\bar{W}_{p_2 r_1}(\sigma''_i, \sigma'_i)}. \tag{2.13}$$

Let us fix the rapidity variables r_1, r_2 and use more simple notations T_{p_1} and \bar{T}_{p_2} .

The partition function is

$$Z = Tr(T_{p_1} \bar{T}_{p_2})^{L/2}. \tag{2.14}$$

In the next sections we derive several functional relations between T_{p_1} and \bar{T}_{p_2} .

We can construct two R -matrices from the weights $\bar{W}_{pq}(\alpha, \beta)$. Define (see figure 3)

$$\bar{S}_{\gamma, \alpha}^{\delta, \beta}(p_1, p_2; r_2, r_1) = \frac{\bar{W}_{r_2 p_1}(\gamma, \alpha) \bar{W}_{p_1 r_1}(\alpha, \delta) \bar{W}_{p_2 r_2}(\beta, \gamma)}{\bar{W}_{p_2 r_1}(\beta, \delta)}. \tag{2.15}$$

This R -matrix satisfies the Yang–Baxter equation

$$\begin{aligned} &\bar{S}_{12}(p_1, p_2; q_1, q_2) \bar{S}_{13}(p_1, p_2; r_1, r_2) \bar{S}_{23}(q_1, q_2; r_1, r_2) \\ &= \bar{S}_{23}(q_1, q_2; r_1, r_2) \bar{S}_{13}(p_1, p_2; r_1, r_2) \bar{S}_{12}(p_1, p_2; q_1, q_2). \end{aligned} \tag{2.16}$$

It leads us to the following commutation relation for the product of the transfer matrices $\bar{T}_{p_2} T_{p_1}$:

$$\bar{T}_{p_2} T_{p_1} \bar{T}_{q_2} T_{q_1} = \bar{T}_{q_2} T_{q_1} \bar{T}_{p_2} T_{p_1} \tag{2.17}$$

where p_1, p_2, q_1, q_2 are four points on the curve Γ .

Similarly, one can introduce the second R -matrix S (see [10], for example) and prove another commutation relation

$$T_{p_1} \bar{T}_{p_2} T_{q_1} \bar{T}_{q_2} = T_{q_1} \bar{T}_{q_2} T_{p_1} \bar{T}_{p_2}. \tag{2.18}$$

However, all commutation relations between the transfer matrices are the consequence of the only equation which is called the ‘star–star’ relation [11, 24]. To write it down define two ‘stars’ (see figures 4 and 5)

$$W_{r_2 r_1}^{p_1 p_2}(\alpha, \beta, \gamma, \delta) = \sum_{\sigma} \frac{\bar{W}_{p_1 r_1}(\alpha, \sigma) \bar{W}_{p_2 r_2}(\gamma, \sigma) \bar{W}_{r_2 p_1}(\sigma, \beta)}{\bar{W}_{p_2 r_1}(\delta, \sigma)} \tag{2.19}$$

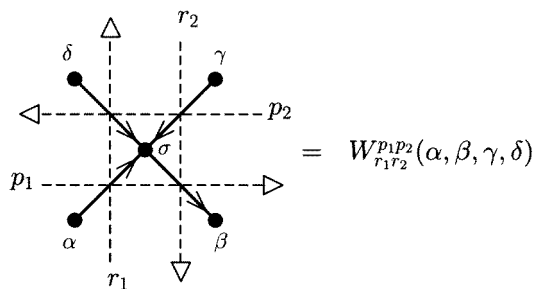


Figure 4. The 'star' weight $W_{r_1 r_2}^{p_1 p_2}(\alpha, \beta, \gamma, \delta)$.

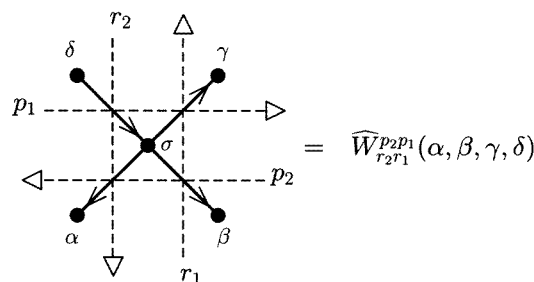


Figure 5. The 'star' weight $\widehat{W}_{r_2 r_1}^{p_2 p_1}(\alpha, \beta, \gamma, \delta)$.

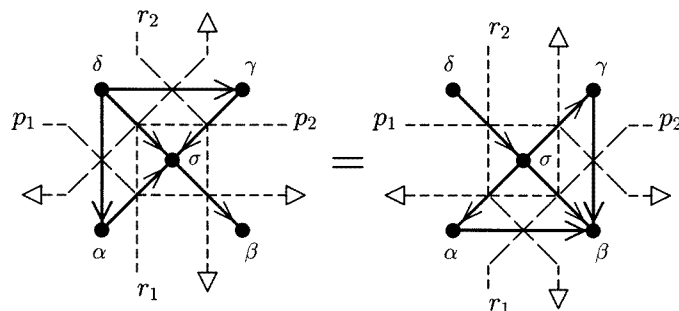


Figure 6. The 'star-star' relation.

and

$$\widetilde{W}_{r_2 r_1}^{p_2 p_1}(\alpha, \beta, \gamma, \delta) = \sum_{\sigma} \frac{\bar{W}_{p_1 r_1}(\sigma, \gamma) \bar{W}_{p_2 r_2}(\sigma, \alpha) \bar{W}_{r_2 p_1}(\delta, \sigma)}{\bar{W}_{p_2 r_1}(\sigma, \beta)}. \tag{2.20}$$

In these notations the star-star relation can be written as (see figure 6)

$$\bar{W}_{p_2 p_1}(\delta, \alpha) \bar{W}_{r_2 r_1}(\delta, \gamma) W_{r_1 r_2}^{p_1 p_2}(\alpha, \beta, \gamma, \delta) \widetilde{W}_{r_2 r_1}^{p_2 p_1}(\alpha, \beta, \gamma, \delta) \bar{W}_{r_2 r_1}(\alpha, \beta) \bar{W}_{p_2 p_1}(\gamma, \beta). \tag{2.21}$$

The Yang-Baxter equation (2.16) can be easily obtained by repeated applications of (2.21).

Now using (2.21) we obtain one more 'commutation' relation between the transfer matrices T_{p_1}, \bar{T}_{p_2} . To do that define the shift operator

$$X_{\sigma_1 \dots \sigma_M}^{\sigma'_1 \dots \sigma'_M} = \prod_{i=1}^M \bar{\delta}_{\sigma_i, \sigma'_{i+1}}. \tag{2.22}$$

Then, using cyclic boundary conditions in a horizontal direction and relation (2.21), we obtain

$$T_{p_1} \bar{T}_{p_2} U^{(d)} X^{-1} = U^{(d)} X^{-1} \bar{T}_{p_2} T_{p_1} \quad (2.23)$$

where $U^{(d)}$ is the diagonal matrix independent of p_1, p_2

$$[U^{(d)}]_{\sigma_1, \dots, \sigma_M}^{\sigma'_1, \dots, \sigma'_M} = \prod_{i=1}^M \bar{\delta}_{\sigma_i, \sigma'_i} \bar{W}_{r_2, r_1}(\sigma_i, \sigma_{i+1}). \quad (2.24)$$

3. Functional relations at $q^2 = -1$

. Here we consider only the case $q^2 \equiv \omega = -1$. The product of the transfer matrices $T_{p_1} \bar{T}_{p_2}$ can be easily rewritten in terms of ‘star’ weights (2.19)

$$T_{p_1} \bar{T}_{p_2} = \prod_{i=1}^M W_{r_1 r_2}^{p_1 p_2}(\sigma_i, \sigma_{i+1}, \sigma'_i, \sigma'_i). \quad (3.1)$$

We can specify the horizontal rapidities in such a way that the left and right ‘gauge’ factors $\bar{W}_{p_1, p_2}(\alpha, \beta)$ in (2.21) become degenerate. Namely, there are two simplest choices to do that

$$p_2 = \tau^\lambda(p) \quad p_1 = p \quad \lambda = 0, 1 \quad (3.2)$$

where τ is defined in (2.3) and the case $\lambda = 0$ corresponds just to $p_2 = p_1 = p$.

First let us set $p_2 = p_1 = p$. Then using (2.21) and the explicit form of $\bar{W}_{p_1 p_2}(\alpha, \beta)$ it is easy to see that we have the only non-zero matrix elements for $W_{r_1 r_2}^{pp}(\alpha, \beta, \gamma, \alpha)$ if $\beta = \gamma$. Similarly setting $p_2 = \tau(p_1)$ we obtain that the only non-zero matrix elements for $\bar{W}_{r_1 r_2}^{p\tau(p)}(\alpha, \beta, \beta, \delta)$ are when $\alpha = \delta$.

Therefore, we obtain for $T_p \bar{T}_{\tau^\lambda(p)}$ that if $\sigma_I = \sigma'_I$ for some $I = 1, \dots, M$ then

$$[T_p \bar{T}_{\tau^\lambda(p)}]_{\sigma_1, \dots, \sigma_M}^{\sigma'_1, \dots, \sigma'_M} = 0 \quad \text{if } \sigma_J \neq \sigma'_J \quad \text{for some } J \neq I, \quad J = 1, \dots, M. \quad (3.3)$$

As a result we can split $T_p \bar{T}_{\tau^\lambda(p)}$ into ‘diagonal’ and ‘non-diagonal’ parts. To do this explicitly consider two 4×4 projectors

$$P_{\gamma, \delta}^+ = \frac{1}{4}, \quad P_{\gamma, \delta}^- = \delta_{\gamma, \delta} - \frac{1}{4} \quad \gamma, \delta = 1, \dots, 4. \quad (3.4)$$

Now let us consider R -matrix (2.15) as 4×4 matrix with respect to horizontal indices with fixed vertical ones and denote it as $\bar{S}_{p_1 p_2}(\alpha, \beta)$. Then we have

$$P^+ \bar{S}_{pp}(\alpha, \beta) P^+ = P^+ \bar{S}_{pp}(\alpha, \beta) \quad P^- \bar{S}_{pp}(\alpha, \beta) P^- = \bar{S}_{pp}(\alpha, \beta) P^- \quad (3.5)$$

$$P^+ \bar{S}_{p\tau(p)}(\alpha, \beta) P^+ = \bar{S}_{p\tau(p)}(\alpha, \beta) P^+ \quad P^- \bar{S}_{p\tau(p)}(\alpha, \beta) P^- = P^- \bar{S}_{p\tau(p)}(\alpha, \beta). \quad (3.6)$$

As a consequence we have the following decomposition of the product $T_p \bar{T}_{\tau^\lambda(p)}$:

$$T_p \bar{T}_{\tau^\lambda(p)} = Tr(P^+ \bar{S}_{p\tau^\lambda(p)}(\alpha, \beta) P^+)^M + Tr(P^- \bar{S}_{p\tau^\lambda(p)}(\alpha, \beta) P^-)^M. \quad (3.7)$$

It is not difficult to check that

$$Tr(P^+ \bar{S}_{p\tau^\lambda(p)}(\alpha, \beta) P^+)^M = \Phi_{p, r_{\lambda+1}}^M I \quad (3.8)$$

where I being the identity matrix $4^M \times 4^M$, Φ_{pq} is defined in (2.11) and all subscripts should be considered *modulo 2*. Denote the second term in (3.7) as

$$T^{(\lambda)}(p; r_{1+\lambda}, r_{2+\lambda}) = Tr(P^- \bar{S}_{p\tau^\lambda(p)}(\alpha, \beta) P^-)^M. \quad (3.9)$$

The transfer matrix (3.9) has non-zero matrix elements only if $\sigma_I \neq \sigma'_I, \forall I = 1, \dots, M$.

It is not difficult to construct local ‘fused’ weights for the transfer matrix $T^{(\lambda)}(p; r_{1+\lambda}, r_{2+\lambda})$. First let us write the following decomposition for P^- :

$$P_{\gamma,\delta}^- = \sum_{i=1}^3 c(i, \gamma)c(i, \delta) \quad c(i, \alpha) = \frac{1}{2}\omega^{i\alpha_1 + \frac{1}{2}i(i-1)\alpha_2} \quad i = 1, 2, 3, \quad \alpha \in Z_2 \times Z_2. \tag{3.10}$$

Now consider the family of ‘fused’ L -operators which act in the tensor product of the auxiliary space C^3 and the quantum space C^4 :

$$L_{ij}^{(\lambda)}(p; r_1, r_2) = \Lambda_{p;r_1,r_2} \sum_{\gamma,\delta \in Z_2 \times Z_2} c(i, \gamma)c(j, \delta) \bar{S}_{\gamma,\alpha}^{\delta,\beta}(p, \tau^\lambda(p); r_2, r_1) \tag{3.11}$$

$$\Lambda_{p;r_1,r_2} = \frac{1}{4} \frac{h_3^-(r_1)^2 h_3^+(p)^2 - h_3^-(p)^2 h_3^+(r_1)^2}{h_3^-(r_2)^2 h_3^+(p)^2 - h_3^-(p)^2 h_3^+(r_2)^2} \prod_{i=1}^3 (h_i^-(r_2)h_i^+(p) + h_i^-(p)h_i^+(r_2)) \tag{3.12}$$

and $\bar{S}_{\gamma,\alpha}^{\delta,\beta}(p_1, p_2; r_2, r_1)$ is defined in (2.15).

Introduce the transfer matrix $t^{(\lambda)}(p; r_1, r_2)$ constructed from the ‘fused’ L -operators $L_{ij}^{(\lambda)}(p; r_1, r_2)$ over the auxiliary space C^3

$$t^{(\lambda)}(p; r_1, r_2)_{\sigma_1, \dots, \sigma_M}^{\sigma_1'', \dots, \sigma_M''} = \sum_{i_1} \dots \sum_{i_M} \prod_{\alpha=1}^M [L_{i_\alpha, i_{\alpha+1}}^{(\lambda)}(p; r_1, r_2)]_{\sigma_\alpha}^{\sigma_\alpha''}. \tag{3.13}$$

It is easy to check that two transfer matrices $T^{(\lambda)}(p; r_1, r_2)$ in (3.9) and $t^{(\lambda)}(p; r_1, r_2)$ are related as follows:

$$T^{(\lambda)}(p; r_1, r_2) = \frac{1}{\Lambda_{p;r_1,r_2}^M} t^{(\lambda)}(p; r_1, r_2). \tag{3.14}$$

Therefore, the problem of calculating eigenvalues for the product of two transfer matrices $T_p \bar{T}_{\tau^\lambda(p)}$ in (3.7) is reduced to a calculation of the eigenvalues for the transfer matrix $t^{(\lambda)}(p; r_1, r_2)$ in (3.13).

Let us give explicit formulae for the matrix elements of $L_{ij}^{(\lambda)}(p; r_1, r_2)$. We have

$$L_{ii}^{(\lambda)}(p; r_1, r_2) = u_{i+1}^+(p; r_1, r_2)X_{i+1} + u_{i-1}^-(p; r_1, r_2)X_{i-1} \tag{3.15}$$

$$L_{i,i+1}^{(\lambda)}(p; r_1, r_2) = Z_{i,i+1}(v_{i+1-\lambda}^{(\lambda)}(p; r_1, r_2)X_{i+1-\lambda} + w_{i-1}^{(1-\lambda)}(p; r_1, r_2)X_{i-1}) \tag{3.16}$$

$$L_{i,i-1}^{(\lambda)}(p; r_1, r_2) = Z_{i,i-1}(v_{i-1+\lambda}^{(1-\lambda)}(p; r_1, r_2)X_{i-1+\lambda} + w_{i+1}^{(\lambda)}(p; r_1, r_2)X_{i+1}) \tag{3.17}$$

where

$$u_i^\pm(p; r_1, r_2) = \frac{h_i^\pm(r_2)}{h_i^\pm(r_1)} \prod_{\alpha=1}^3 h_\alpha^\mp(p)h_\alpha^\pm(r_1) \tag{3.18}$$

$$v_i^{(0)}(p; r_1, r_2) = h_i^\mp(p)h_i^\pm(r_2)h_{i+1}^-(r_1)h_{i+1}^+(p)h_{i-1}^-(p)h_{i-1}^+(r_1) \tag{3.19}$$

$$w_i^{(0)}(p; r_1, r_2) = h_i^\mp(p)h_i^\pm(r_2)h_{i+1}^\mp(r_1)h_{i+1}^\pm(p)h_{i-1}^\mp(r_1)h_{i-1}^\pm(p). \tag{3.20}$$

The 4×4 matrices X_i and $Z_{i,j}$ act in $Z_2 \times Z_2$ and have the following matrix elements:

$$\langle \alpha | X_1 | \beta \rangle = \delta_{\alpha_1, \beta_1} \delta_{\alpha_2, \beta_2 - 1} \quad \langle \alpha | X_2 | \beta \rangle = \delta_{\alpha_1, \beta_1 - 1} \delta_{\alpha_2, \beta_2} \tag{3.21}$$

$$\langle \alpha | Z_1 | \beta \rangle = \omega^{\alpha_2} \delta_{\alpha_1, \beta_1} \delta_{\alpha_2, \beta_2} \quad \langle \alpha | Z_2 | \beta \rangle = \omega^{\alpha_1} \delta_{\alpha_1, \beta_1} \delta_{\alpha_2, \beta_2} \tag{3.22}$$

$$X_1 X_2 X_3 = 1 \quad Z_3 = 1 \quad Z_{ij} = Z_i Z_j^{-1} \quad i \neq j = 1, 2, 3. \tag{3.23}$$

They satisfy the usual relations

$$X_i Z_{jk} = \omega^{\delta_{ij} - \delta_{ik}} Z_{jk} X_i \quad Z_{ij} Z_{jk} Z_{ki} = 1. \tag{3.24}$$

Actually, it is convenient to introduce the ‘gauge-transformed’ L -operators

$$\hat{L}_{ij}^{(\lambda)}(x_p; r_1, r_2) = \left[\prod_{i=1}^3 h_i^+(p) \right]^{-1} [\xi_i(p)]^{(-1)^i} L_{ij}^{(\lambda)}(p; r_1, r_2) [\xi_j(p)]^{(-1)^{1+i}} \tag{3.25}$$

where

$$\xi(p) \equiv (\xi_1(p), \xi_2(p), \xi_3(p)) = (x_3(p), 1, x_1^{-1}(p)) \tag{3.26}$$

and $x_p, x_i(p)$ are defined in (2.12).

It is easy to see that L -operators (3.25) depend on the rapidity variable p only via the combination x_p . Therefore, we have

$$t^{(\lambda)}(p; r_1, r_2) = \left(\prod_{i=1}^3 h_i^+(p) \right)^M t^{(\lambda)}(x_p; r_1, r_2) \tag{3.27}$$

where $t^{(\lambda)}(x_p; r_1, r_2)$ is the transfer matrix constructed from the L -operators (3.25) and is obviously just a polynomial in the variable x_p of the degree M .

It follows that we can consider the parameters $x_p, h_i^\pm(r_1), h_i^\pm(r_2)$ as independent ones and do not take into account the surface equations (2.1).

L -operators (3.25) satisfy the following Yang–Baxter equation:

$$\sum_{i_2, j_2} R_{i_1, i_2; j_1, j_2}^{(\lambda)}(x/y) \hat{L}_{i_2 i_3}^{(\lambda)}(x^2) \hat{L}_{j_2 j_3}^{(\lambda)}(y^2) = \sum_{i_2, j_2} \hat{L}_{i_1 j_2}^{(\lambda)}(y^2) \hat{L}_{i_2 i_3}^{(\lambda)}(x^2) R_{i_2, i_3; j_2, j_3}^{(\lambda)}(x/y) \tag{3.28}$$

where we omit a dependence on r_1, r_2 in $\hat{L}^{(\lambda)}(x; r_1, r_2)$ and $R_{i_1, i_2; j_1, j_2}^{(\lambda)}(x)$ coincides with the deformed trigonometric $sl(3)$ R -matrix. Namely, let us assume for a moment that q is arbitrary and introduce the R -matrix [25]

$$R_{i_1, i_2; j_1, j_2}(x, q, \rho) = \delta_{i_1 i_2} \delta_{j_1 j_2} \delta_{i_1 j_1} (q - 1)(x + x^{-1} q^{-1}) + \delta_{i_1 i_2} \delta_{j_1 j_2} \rho_{i_1 j_1} (x - x^{-1}) + \delta_{i_1 j_2} \delta_{i_2 j_1} (1 - \delta_{i_1 j_1}) \sigma_{i_1 i_2}(x) \tag{3.29}$$

where

$$\rho_{ii} = \rho_{ij} \rho_{ji} = 1 \quad \sigma_{ij} \equiv \begin{cases} 0 & i = j \\ (q - q^{-1})x & i < j \\ (q - q^{-1})x^{-1} & i > j. \end{cases} \tag{3.30}$$

Then

$$R_{i_1, i_2; j_1, j_2}^{(\lambda)}(x) = R_{i_1, i_2; j_1, j_2}(x^{(-1)^\lambda}, q, \rho) \tag{3.31}$$

with

$$q = i \quad \rho_{j, j+1} = \rho_{j+1, j}^{-1} = i^{(-1)^{\lambda+1}} \quad j = 1, 2, 3. \tag{3.32}$$

Now using a fusion technique for the R -matrix (3.29) we can construct functional equations for the transfer matrix $t^{(\lambda)}(x_p; r_1, r_2)$. Actually, this procedure is quite similar to the fusion for the $sl(3)$ trigonometric R -matrix, so we do not give a detailed derivation of the functional equations.

We can define L -operators related to the ‘antisymmetric’ representation

$$\bar{L}_{ij}^{(\lambda)}(x; r_1, r_2) = \frac{1}{\phi_1^-(x; r_{1+\lambda})} [\hat{L}_{km}^{(\lambda)}((-1)^\lambda x; r_1, r_2) \hat{L}_{ln}^{(\lambda)}((-1)^{1+\lambda} x; r_1, r_2) - (-1)^{\lambda+\delta_{i_2}} \hat{L}_{lm}^{(\lambda)}((-1)^\lambda x; r_1, r_2) \hat{L}_{kn}^{(\lambda)}((-1)^{1+\lambda} x; r_1, r_2)] \tag{3.33}$$

where indices $\{i, k, l\}$ and $\{j, m, n\}$ are even permutations of $\{1, 2, 3\}$ and we define

$$\phi_1^\pm(x; r) = \left(\prod_{i=1}^3 h_i^-(r) \pm x \prod_{i=1}^3 h_i^+(r) \right). \tag{3.34}$$

Let $\bar{t}^{(\lambda)}(x; r_1, r_2)$ be the transfer matrix constructed from the L -operators (3.33).

Omitting a dependence of $t^{(\lambda)}(x_p; r_1, r_2)$, $\bar{t}^{(\lambda)}(x_p; r_1, r_2)$ on r_1, r_2 one can show that the following system of functional equations holds:

$$t^{(\lambda)}(x_p)t^{(\lambda)}(\omega x_p) = \Phi_0(x_p; r_{1+\lambda, 2+\lambda}) + \phi_1^-(x_p; r_{1+\lambda})^M \bar{t}^{(\lambda)}(x_p) + \phi_1^+(x_p; r_{1+\lambda})^M \bar{t}^{(\lambda)}(\omega x_p) \quad (3.35)$$

$$\bar{t}^{(\lambda)}(x_p)\bar{t}^{(\lambda)}(\omega x_p) = \Phi_0(x_p; r_{2+\lambda, 1+\lambda}) + \phi_1^-(x_p; r_{2+\lambda})^M t^{(\lambda)}(x_p) + \phi_1^+(x_p; r_{2+\lambda})^M t^{(\lambda)}(\omega x_p) \quad (3.36)$$

with $\phi_1^\pm(x_p; r)$ defined in (3.34) and

$$\Phi_0(x_p; r', r) = \lambda_1^M + \lambda_2^M + \lambda_3^M \quad (3.37)$$

where λ_i , $i = 1, 2, 3$ are three roots of the following cubic equation:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \quad (3.38)$$

$$a = x_p^2 \prod_{i=1}^3 h_i^+(r)^2 \sum_{i=1}^3 \frac{h_i^+(r')^2}{h_i^+(r)^2} - \prod_{i=1}^3 h_i^-(r)^2 \sum_{i=1}^3 \frac{h_i^-(r')^2}{h_i^-(r)^2} \quad (3.39)$$

$$b = \prod_{i=1}^3 \frac{h_i^+(r)^2}{[h_i^+(r')]^2} \left(\prod_{i=1}^3 \frac{h_i^-(r)^2}{h_i^+(r)^2} - x_p^2 \right) \left[\prod_{i=1}^3 \frac{h_i^-(r')^2}{h_i^+(r')^2} \sum_{i=1}^3 \frac{h_i^-(r)^2}{h_i^-(r')^2} - x_p^2 \sum_{i=1}^3 \frac{h_i^+(r)^2}{h_i^+(r')^2} \right] \quad (3.40)$$

$$c = \left(x_p^2 \prod_{i=1}^3 h_i^+(r')^2 - \prod_{i=1}^3 h_i^-(r')^2 \right) \left(x_p^2 \prod_{i=1}^3 h_i^+(r)^2 - \prod_{i=1}^3 h_i^-(r)^2 \right)^2. \quad (3.41)$$

A system of functional equations similar (3.35), (3.36) for ‘factorized’ $sl(3)$ L -operators has been obtained in [24] and discussed in [11]. However, for that case there is no Z_3 symmetry with respect to the indices of L -operators and as a result, the functional relations will have a slightly more complicated form and involve explicitly shift operators in the ‘quantum’ space.

It is easy to see that the system of functional equations (3.35), (3.36) is invariant under the replacement $r_1 \rightarrow r_2$, $t^{(\lambda)}(x_p; r_1, r_2) \rightarrow \bar{t}^{(\lambda)}(x_p; r_1, r_2)$. However, it is not true in general that all eigenvalues of the transfer matrices $t^{(\lambda)}(x_p)$, $\bar{t}^{(\lambda)}(x_p)$ satisfy $\bar{t}^{(\lambda)}(x_p; r_1, r_2) = t^{(\lambda)}(x_p; r_2, r_1)$. The spectrum of $t^{(\lambda)}(x_p)$, $\bar{t}^{(\lambda)}(x_p)$ also involves ‘non-symmetric’ solutions of (3.35), (3.36).

4. The homogeneous case $r_1 = r_2$

A great simplification occurs if we restrict ourselves to the case of the completely homogeneous model $r_1 = r_2 = r$. In this case all roots in (3.38) coincide and the function $\Phi_0(x_p; r, r)$ has a very simple form. Let us define

$$x_r = \prod_{i=1}^3 \frac{h_i^-(r)}{h_i^+(r)}. \quad (4.1)$$

It is easy to see that for this case the transfer matrices $t(x_p; r, r)$ and $\bar{t}(x_p; r, r)$ are effectively the functions of a single variable $x = x_p/x_r$. Namely,

$$t(x_p; r, r) = \left(x_r \prod_{i=1}^3 h_i^+(r) \right)^M t(x_p/x_r) \quad \bar{t}(x_p; r, r) = \left(x_r \prod_{i=1}^3 h_i^+(r) \right)^M \bar{t}(x_p/x_r). \quad (4.2)$$

Then, omitting the superscript (λ) , the system of functional equations (3.35), (3.36) can be rewritten as follows:

$$\begin{aligned} t(x)t(-x) &= 3(1-x^2)^M + (1-x)^M \bar{t}(x) + (1+x)^M \bar{t}(-x) \\ \bar{t}(x)\bar{t}(-x) &= 3(1-x^2)^M + (1-x)^M t(x) + (1+x)^M t(-x) \end{aligned} \quad (4.3)$$

where $t(x)$ and $\bar{t}(x)$ are polynomials in x of the degree M .

Let us define

$$u(x) = t(x) + (1+x)^M \quad \bar{u}(x) = \bar{t}(x) + (1+x)^M. \quad (4.4)$$

Then we have from (4.3)

$$u(x)u(-x) = \bar{u}(x)\bar{u}(-x) \quad (4.5)$$

$$u(x)u(-x) = (1-x)^M(u(x) + \bar{u}(x)) + (1+x)^M(u(-x) + \bar{u}(-x)). \quad (4.6)$$

The solution of (4.5) can be written as follows:

$$u(x) = a(x)b(x) \quad \bar{u}(x) = a(-x)b(x) \quad a(x) = \prod_{i=1}^k (u_i - x) \quad k = 0, \dots, M \quad (4.7)$$

$u_i, i = 1, \dots, k$ —the roots of the polynomial $a(x)$ of the degree k and $b(x)$ is the polynomial in x of the degree $M - k$.

Then we have from (4.5), (4.6)

$$c(x)c(-x) = (1-x^2)^M(a(x) + a(-x))^2 \quad (4.8)$$

$$c(x) = a(x)a(-x)b(x) - (1+x)^M(a(x) + a(-x)) \quad (4.9)$$

where $c(x)$ is the polynomial in x of the degree $M + k$.

In principle it is straightforward to solve (4.8), (4.9) in terms of u_i . First we need to find the roots of the polynomial $a(x) + a(-x)$:

$$a(x) + a(-x) = \lambda \prod_{i=1}^{\lfloor k/2 \rfloor} (v_i^2 - x^2) \quad (4.10)$$

where $\lfloor k/2 \rfloor$ is the integer part of $k/2$.

Now a general solution of (4.8) has the following form:

$$c(x) = \lambda(1-x)^n(1+x)^{M-n} \prod_{i=1}^{l_1} (v_i - x) \prod_{i=l_1+1}^{\lfloor k/2 \rfloor} (v_i + x) \prod_{i=1}^{l_2} (v_i - x) \prod_{i=l_2+1}^{\lfloor k/2 \rfloor} (v_i + x) \quad (4.11)$$

$$n = 0, \dots, M, \quad 0 \leq l_1 \leq l_2 \leq \lfloor k/2 \rfloor.$$

The last step is to substitute (4.7), (4.10) and (4.11) into (4.9) and demand that $b(x)$ should be a polynomial in x . It gives us a closed system of equations on u_i and v_i .

As a result we come to the system of equations on u_i and v_i :

$$\prod_{j=1}^k \frac{u_j - v_i}{u_j + v_i} = -1 \quad i = 1, \dots, \lfloor k/2 \rfloor$$

$$\prod_{j=1}^k (u_j + u_i) = \prod_{j=1}^{\lfloor k/2 \rfloor} (v_j^2 - u_i^2) \quad i = 1, \dots, k \quad (4.12)$$

$$\prod_{j=1}^{l_1} \frac{v_j - u_i}{v_j + u_i} \prod_{j=l_2+1}^{\lfloor k/2 \rfloor} \frac{v_j + u_i}{v_j - u_i} = - \left(\frac{1 + u_i}{1 - u_i} \right)^n \quad n = 0, \dots, M, l_1 \quad l_2 = 0, \dots, \lfloor k/2 \rfloor. \quad (4.13)$$

It is not difficult to solve (4.12), (4.13) for $k = 0, 1$ and to obtain some eigenvalues of the transfer matrix $T_p \bar{T}_p$ at $r_1 = r_2$ for any M . However, in general the system (4.12) and (4.13) is a transcendental system of equations and cannot be solved explicitly for any values of k .

Therefore, (4.7)–(4.13) together with (4.1), (4.2) and (4.4) give the solution for the spectrum of the transfer matrix $T_p \bar{T}_p$ in the homogeneous case $r_1 = r_2$. However, we should say that this case is quite restrictive and, in fact, all dependence on the vertical rapidity r and matrices K_{ij} in (2.1) can be absorbed by a redefinition of the parameter x_p .

5. The Bethe ansatz technique for ‘fused’ L -operators

In the previous section we considered the solution to the functional relations (3.35), (3.36) for the case $p_1 = p_2 = p$, $r_1 = r_2 = r$. However, a direct Bethe ansatz method can also be developed for this case. Let us make some transformations of the L -operators given by (3.15)–(3.17) (here we consider only the case $\lambda = 0$). Define the L' -operators:

$$L'_{ij} = \kappa \xi_i(p) / \xi_i(r) L_{ij}^{(0)}(p; r, r) \xi_j(r) / \xi_j(p) \quad (5.1)$$

where $\xi(p)$ is defined in (3.26),

$$\kappa = (h^+(p)h^-(p)h^+(r)h^-(r))^{-1/2} \quad h^\pm(p) = \prod_{i=1}^3 h_i^\pm(p). \quad (5.2)$$

The transfer matrices for $L^{(0)}$ - and L' -operators differ only by the scalar factor κ^M . Then all matrix elements of L' are the functions of a single parameter $x = (x_p/x_r)^{1/2}$. To obtain the simplest form of L' let us make the following equivalence transformation:

$$\bar{L}_{ij} = C L'_{ij} C^{-1} \quad C = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (5.3)$$

Then we have for the matrix elements of the \bar{L} -operators

$$\bar{L}_{ii}(\alpha, \beta) = \{x\}(\delta_{\alpha,0}\delta_{\beta,0} - \delta_{\alpha,i}\delta_{\beta,i}) + [x](\delta_{\alpha,i+1}\delta_{\beta,i+1} - \delta_{\alpha,i-1}\delta_{\beta,i-1}) \quad (5.4)$$

$$\bar{L}_{ij}(\alpha, \beta) = 2x^{\epsilon_{ij}}(\delta_{\alpha,k}\delta_{\beta,0} - \delta_{\alpha,j}\delta_{\beta,i}) \quad (5.5)$$

where $\{x\} = x + 1/x$, $[x] = x - 1/x$, the indices $i - 1, i, i + 1$ in (5.4) are defined by the cyclic permutation of 1, 2, 3 while i, j, k in (5.5) is any permutation of 1, 2, 3 and

$$\epsilon_{ij} = \begin{cases} 1 & i < j \\ -1 & i > j. \end{cases} \quad (5.6)$$

In (5.4), (5.5) we imply that the enumeration of rows and columns is given by (0, 1, 2, 3).

From (5.4), (5.5) we can easily observe that: (i) the representation matrices of the diagonal L -operators are diagonal and for all of them the component (0, 0) is equal to $\{x\}$; (ii) the non-diagonal L -operators have zeros in the 0-rows.

Therefore, we can conclude that for the transfer matrix

$$\bar{T}_{\alpha_1, \dots, \beta_M}^{\beta_1, \dots, \beta_M} = \sum_{i_1, \dots, i_M} \bar{L}_{i_1 i_2}(\alpha_1, \beta_1) \dots \bar{L}_{i_M i_1}(\alpha_M, \beta_M) \quad (5.7)$$

all matrix elements $\bar{T}_{\alpha_1, \dots, \alpha_M}^{j_1, \dots, j_M} = 0$, for which there is at least one 0-component among the set $\alpha_1, \dots, \alpha_M$ and all indices j_k run the values (1, 2, 3). So, the transfer matrix (5.7) has the block-down-triangular form.

For example, the simplest block has a dimension 1:

$$\bar{T}_{0, \dots, 0}^{0, \dots, 0} = 3\{x\}^M. \quad (5.8)$$

Further there are M blocks 3×3 of the form:

$$\bar{T}_{0, \dots, 0, i_k, 0, \dots, 0}^{0, \dots, 0, j_k, 0, \dots, 0} = \{x\}^{M-1} T_{i_k}^{(1)j_k} \quad (5.9)$$

where $T^{(1)}$ is the one-site transfer matrix for the ‘reduced’ L -operators which can be obtained from \bar{L} by removing all 0-components, i.e.

$$L_{ij}(i_1, j_1) = \bar{L}_{ij}(i_1, j_1) \quad i_1, j_1 = (1, 2, 3). \tag{5.10}$$

The next $M(M - 1)/2$ blocks in the transfer matrix have the dimension $3^2 \times 3^2$:

$$\bar{T}_{0, \dots, 0, i_{k1}, 0, \dots, i_{k2}, 0, \dots, 0}^{0, \dots, 0, j_{k1}, 0, \dots, j_{k2}, 0, \dots, 0} = \{x\}^{M-2} T_{i_{k1}, i_{k2}}^{(2) j_{k1}, j_{k2}} \tag{5.11}$$

where $T^{(2)}$ is the two-site transfer matrix for the ‘reduced’ L -operators. In the n th step we have C_M^n blocks obtained from the n -site transfer matrices. In the last step we have one block: $T_{i_1, \dots, i_M}^{(M) j_1, \dots, j_M}$. As a result we obtain a decomposition of the initial 4^M -dimensional space into the direct sum of the subspaces $\sum_{n=0}^M C_M^n 3^n$. Therefore, the initial problem of calculation of the spectrum for the transfer matrix (5.7) has been reduced to the spectral problem for the transfer matrices $T^{(n)}$.

It is easy to see that the L -operators given by (5.10) coincide with the R -matrix (3.29) for the $sl(3)$ model. Hence, one can use the standard nested Bethe ansatz technique [20] for this model. We give here only the final result for the eigenvalues for the deformed $sl(3)$ model with the arbitrary deformation parameters q and $\rho_{i,j}$:

$$\Lambda(x) = \frac{[qx]^n}{\rho_{1,3}^b \rho_{2,3}^{a-b}} \prod_{i=1}^a w\left(\frac{y_i}{x}\right) + [x]^n \left[\frac{\rho_{2,3}^{n-a}}{\rho_{1,2}^b} \prod_{i=1}^a w\left(\frac{x}{y_i}\right) \prod_{k=1}^b w\left(\frac{z_k}{x}\right) + \frac{\rho_{1,3}^{n-a}}{\rho_{1,2}^{b-a}} \prod_{k=1}^b w\left(\frac{x}{z_k}\right) \right] \tag{5.12}$$

where $0 \leq b \leq a \leq n$, $w(x) = \frac{[qx]}{[x]}$ and the two sets of parameters y_1, \dots, y_a and z_1, \dots, z_b should be defined from the system of the Bethe ansatz equations:

$$w(y_i)^n = (-1)^{a-1} \rho_{2,3}^n \rho^{-b} \prod_{j \neq i}^a \frac{[qy_i/y_j]}{[qy_j/y_i]} \prod_{k=1}^b w\left(\frac{z_k}{y_i}\right)$$

$$\prod_{i=1}^a w\left(\frac{z_k}{y_i}\right) = (-1)^{b-1} \left(\frac{\rho_{1,3}}{\rho_{2,3}}\right)^n \rho^a \prod_{l \neq k}^b \frac{[qz_k/z_l]}{[qz_l/z_k]}$$

where $\rho = \rho_{1,2} \rho_{2,3} \rho_{3,1}$. However, we need only a particular case of this solution

$$q = i \quad \rho_{j,j+1} = -\rho_{j+1,j} = -i \quad j = 1, 2, 3. \tag{5.13}$$

In this case the formulae (5.13) become much more simple:

$$w'(y_i)^n = (-1)^{n-a-1} \prod_{k=1}^b w'\left(\frac{z_k}{y_i}\right) \quad \prod_{i=1}^a w'\left(\frac{z_k}{y_i}\right) = (-1)^{n-b-1} \quad w'(x) = \frac{[x]}{[x]}. \tag{5.14}$$

In comparison with equations (4.12), (4.13), equations (5.14) do not contain the redundant solutions. The reason for this can be easily understood. Namely, if we consider another type of L -operators which can be obtained from the initial ones by the transposition of the representation matrices, one can conclude that the transfer matrix for them satisfies the same functional relations (4.3). The resulting equations of the Bethe ansatz for this case can be obtained from (5.13) by the following substitution:

$$x \rightarrow x^{-1} \quad y_i \rightarrow y_i^{-1} \quad z_k \rightarrow z_k^{-1} \quad \rho_{i,j} \rightarrow \rho_{j,i}. \tag{5.15}$$

After this we should fix the parameters of deformation as in (5.13). So, the system of the equations (4.12), (4.13) contains the solutions to Bethe ansatz equations for both cases. Unfortunately, we have failed to find an explicit correspondence between (4.12), (4.13) and (5.14).

6. Discussion

In this paper we have obtained the functional relations for eigenvalues of the transfer matrix of the $sl(3)$ CPM at $q^2 = -1$. In the completely homogeneous case we have also developed a direct Bethe ansatz scheme.

A whole set of functional equations for the usual chiral Potts model has been obtained in [26]. However, the $sl(3)$ case looks much more difficult. In particular, a proper generalization of the Baxter construction of the Q -matrix for the eight-vertex model [5] which works for the usual CPM [6] fails for the $sl(3)$ case. However, the structure of functional equations for the arbitrary N should be governed by a proper generalization of quadratic ‘fusion’ rules [27, 28] which should have the same functional form for the $sl(n)$ case at roots of unity as well. Then it should be possible to define a set of Q -matrices for the $sl(n)$ case. Of course, the boundary conditions for ‘fusion’ rules and analytical properties of solutions at $q^N = -1$ will be extremely complicated. However, even a trigonometric limit of the $sl(n)$ chiral Potts model at $q^N = -1$ is of a great interest, because it corresponds to the n -layer Zamolodchikov model under the proper modification of boundary conditions on a three-dimensional lattice. We hope to address these problems in further publications.

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References

- [1] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M L 1987 *Phys. Lett. A* **123** 219–23
- [2] McCoy B M, Perk J H H, Tang S and Sah C H 1987 *Phys. Lett. A* **125** 9–14
- [3] Baxter R J, Perk J H H and Au-Yang H 1988 *Phys. Lett. A* **128** 138–42
- [4] Au-Yang H and Perk J H H 1989 *Proc. Taniguchi Symp.* vol 19 (Kinokuniya/Academic)
- [5] Baxter R J 1982 *Exactly Soluble Models in Statistical Mechanics* (London: Academic)
- [6] Bazhanov V V and Stroganov Yu G 1990 *J. Stat. Phys.* **59** 799
- [7] Lieb E H 1967 *Phys. Rev.* **162** 162–72
- [8] Bazhanov V V, Kashaev R M, Mangazeev V V and Stroganov Yu G 1991 *Commun. Math. Phys.* **138** 393–408
- [9] Date E, Jimbo M, Miki K and Miwa T 1991 *Commun. Math. Phys.* **137** 133
- [10] Kashaev R M, Mangazeev V V and Nakanishi T 1991 *Nucl. Phys. B* **362** 563
- [11] Bazhanov V V and Baxter R J 1992 *J. Stat. Phys.* **69** 453–85
- [12] Zamolodchikov A B 1980 *Zh. Eksp. Teor. Fiz.* **79** 641–664
Zamolodchikov A B 1980 *JETP* **52** 325–36 (English translation)
- [13] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489–505
- [14] Bazhanov V V and Stroganov Yu G 1982 *Teor. Mat. Fiz.* **52** 105–13
Bazhanov V V and Stroganov Yu G 1982 *Theor. Math. Phys.* **138** 685–91 (English translation)
- [15] Jaekel M T and Maillard J M 1982 *J. Stat. Phys.* **15** 1309
- [16] Baxter R J 1983 *Commun. Math. Phys.* **88** 185
- [17] Kashaev R M, Mangazeev V V and Stroganov Yu G 1993 *Int. J. Mod. Phys. A* **8** 1399–409
- [18] Baxter R J 1986 *Physica D* **18** 321–47
- [19] Baxter R J and Quispel G R W 1990 *J. Stat. Phys.* **58** 411–30
- [20] Kulish P P and Reshetikhin N Yu 1981 *Zh. Eksp. Teor. Fiz.* **80** 214
Babelon O, De Vega H J and Viallet C M 1982 *Nucl. Phys. B* **220** 266
- [21] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393–403
- [22] Kirillov A N and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1587

- [23] Bazhanov V V and Reshetikhin Yu N 1990 *J. Phys. A: Math. Gen.* **23** 1477
- [24] Bazhanov V V and Kashaev R M 1991 Private communication
- [25] Cherednik I V 1980 *Teor. Mat. Fiz.* **43** 117–9
Kulish P P and Sklyanin E K *Zapiski Nauch. Semin. LOMI* **95** 129–60
Perk J H H and Schultz C L 1981 *Phys. Lett. A* **84** 407
- [26] Baxter R J, Bazhanov V V and Perk J H H 1990 *Int. J. Mod. Phys. B* **4** 803–70
- [27] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys. A* **9** 5215–312
- [28] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1997 *Commun. Math. Phys.* **188** 267–304